Early Cosmology

The Schwarzschild solution

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Outline

- History
- The Schwarzschild metric
- Discussion

History

History

- discovered by Karl Schwarzschild in 1915
- „Über das Gravitationsfeld eines Massenpunktes nach der Einstein‘schen Theorie“
- First non-trivial exact solution of Einstein‘s field equations

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 + \frac{1}{1 - \frac{r_s}{r}}\mathrm{d}r^2$$
$$+ r^2\left(\mathrm{d}\theta^2 + \sin\theta \mathrm{d}\varphi^2\right)$$

coordinate
singularity

History

- A lot of wonder about whether or not the singularity at the Schwarzschild radius is physical
- Eddington (1924) and Lemaître (1932) recognise the singularity to be an artefact
- Robertson shows 1939: falling observer passes singularity in finite proper time, yet infinite coordinate time

History

- Kruskal found a coordinate transformation allowing for an analytic continuation to (almost) the entire spacetime
- 1960s: Differential geometry becomes more important in gRT and definitions become more exact
- Definitive identification of singularity at Schwarzschild radius as event horizon

Math
The Schwarzschild metric

The Schwarzschild metric

Empty-space-solution of Einstein's field equations for a stationary spherically symmetric mass distribution

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}$$

Empty space, so no sources from the energy-momentum tensor

$$T_{\mu\nu} = 0$$

The Schwarzschild metric

Objects we need from differential geometry

$g_{\mu\nu}$	metric, describes pace-time
$\Gamma^{\mu}_{\kappa\lambda}$	Christoffel symbols: defined for covariant derivate (keep tensor properties when differentiating)
$R^{\mu}_{\kappa\lambda\alpha}$	Riemann tensor: commutator of covariant derivatives, measure for curvature
$R_{\mu\nu}$	Ricci tensor: contraction of Riemann tensor, also reflects curvature
R	Ricci scalar: contraction of Ricci tensor

The Schwarzschild metric

Choose spherical coordinates

$$x^\mu = (t, r, \theta, \varphi)$$

Due to rotational symmetry and time-independence the line element assumes the form

$$\mathrm{d}s^2 = -A\mathrm{d}t^2 + B\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)$$

functions of the radius, to be determined by the field equations

The Schwarzschild metric

Plug ansatz into Euler-Lagrange formalism and compare with geodesic equations to obtain Christoffel symbols

$$\partial \int g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} \mathrm{d}\sigma = 0$$

$$\ddot{x}^\mu + \Gamma^\mu_{\kappa\lambda} \dot{x}^\lambda \dot{x}^\kappa = 0$$

Only 13 of the possible 64 Christoffel symbols are non-zero.

The Schwarzschild metric

Introduce auxiliary relations

$$\sqrt{-\det(g)} = r^2|\sin\theta|\sqrt{AB}$$

$$\partial_\beta \log\sqrt{-g} = \Gamma^\mu_{\mu\beta}$$

Simplify field equaitons because there is no source

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\alpha_\alpha\right) = 0$$

The Schwarzschild metric

Write down Ricci-Tensor

Off-diagonal elements are zero,
rest gives us…

$$B = \frac{1}{A}$$

Integration constant: we want flat space-time infinitely far away

$$A = 1 - \frac{r_s}{r} \qquad B = \left(1 - \frac{r_s}{r}\right)^{-1}$$

r_s…Schwarzschild radius: integration constant

The Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 + \frac{1}{1 - \frac{r_s}{r}}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin\theta \mathrm{d}\varphi^2\right)$$

$$r_s = \frac{GM}{2c^2}$$

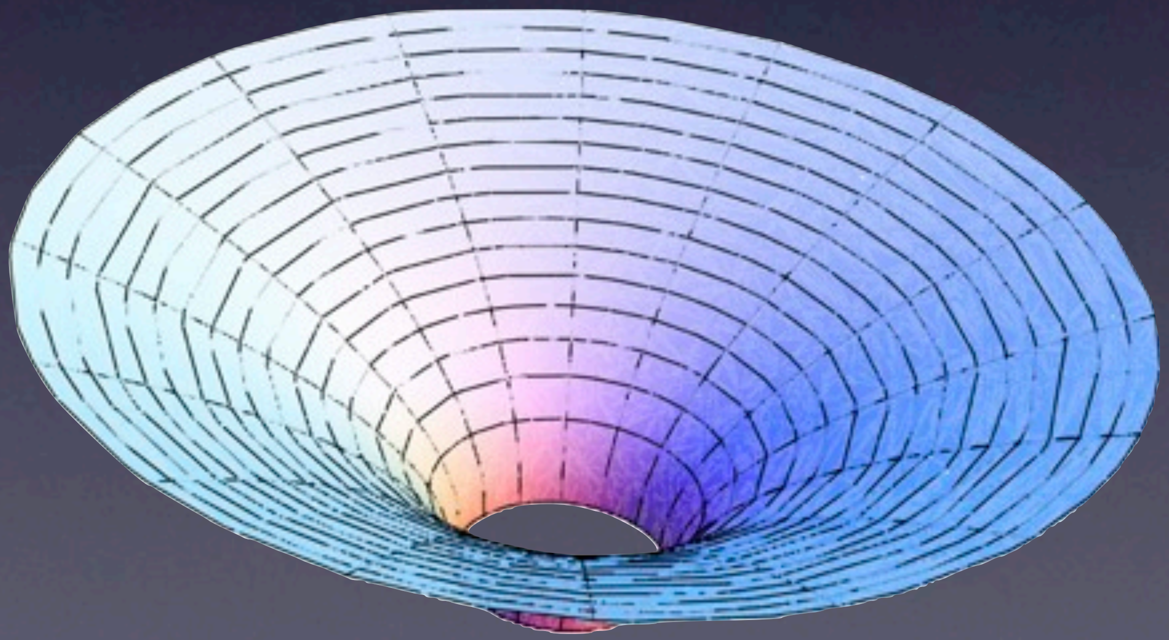
obtained from classical
Newtonia gravitatio for far-field

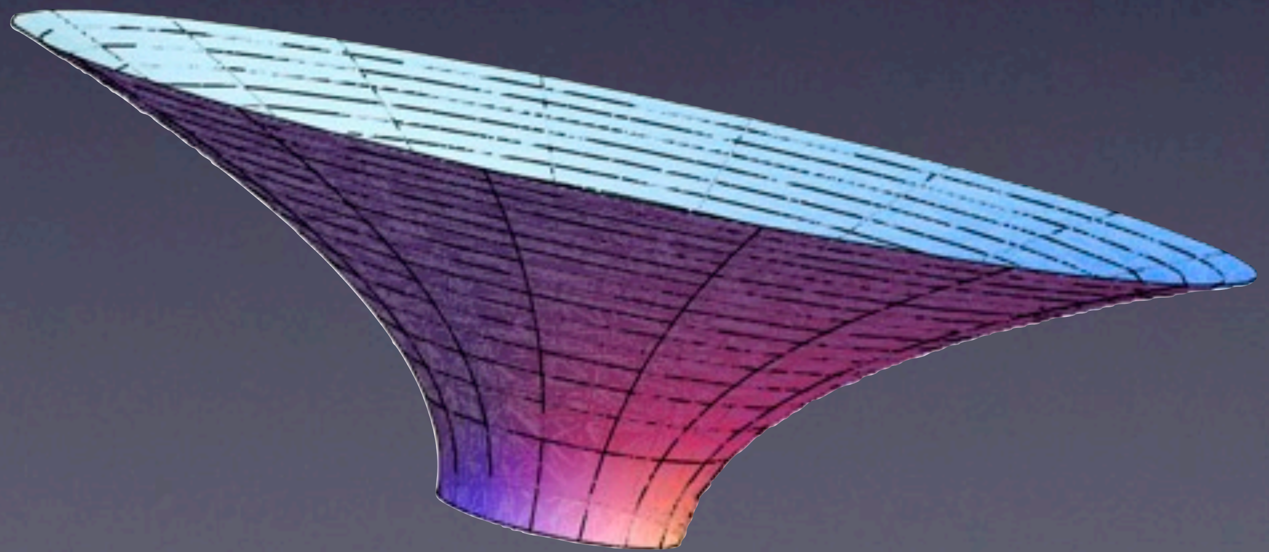
Discussion

Visualisation

Flamm‘s paraboloid: visualise metric as revolution plot for fixed time in equatorial plane with an extra (non-physical) z-dimension

$$z = 2\sqrt{r_s\left(r - r_s\right)}$$





Singularities

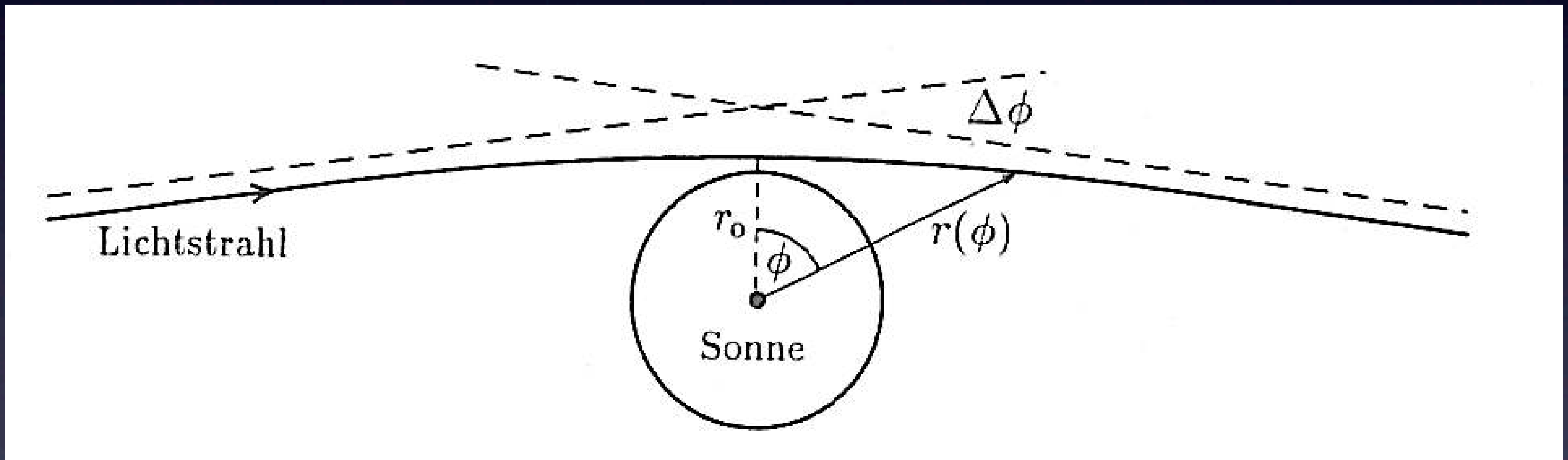
- No problem for ordinary stars: geometric radius is always larger than Schwarzschild radius. E.g. for the sun

$$R = 7 \cdot 10^5 \text{ km} \qquad r_s = 3 \text{ km}$$

- Singularity at origin is physical. It cannot be transformed away. Space-time becomes infinite: „black hole“, Inside: r $\leftrightarrow$ t exchange.

Light deflexion

Light ray is deflected when passing a massive object



First observed in 1919 at a solar eclipse
Modern observations with quasars

Explains phenomenon of gravitational lensing

References

- K. Schwarzschild (1916). "Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie". Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften 7: 189–196.
  English translation: arXiv:physics/9905030.
- G. 't Hooft, An Introduction to General Relativity (Utrecht, 1998)
- E. Kreyszig, Differentialgeometrie (Leipzig, 1957)
- T. Fließbach, Allgemeine Relativitätstheorie (Berlin, 2006)
- http://en.wikipedia.org/wiki/Schwarzschild_metric

Questions

- What does it mean when space and time change their role ($t \leftrightarrow r$ in a black hole)?
- How important are black holes in terms of universe formation? Which role do they play?
- Do we have means of measuring the accumulated mass in a black hole?
- *Steven: Black holes in Quantum-Gravity?*

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