

Saturation of Fleming's Quantum Master Inequality

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Outline

- Derivation of the QMIE
- Saturation of the QMIE
 - General criterion for saturation
 - Conditions to the saturating operator

Stage & Notation

\mathcal{H} Hilbert space

$\langle \cdot, \cdot \rangle$ scalar product in the Hilbert space

$|\cdot|$ norm induced by the scalar product

$\psi, \psi' \in \mathcal{H}$

$|\psi| = |\psi'| = 1$ two normalised state vectors

$A : \mathcal{H} \rightarrow \mathcal{H}, \psi \mapsto A\psi$ linear, continuous

$\langle A \rangle_\psi := \langle \psi, A\psi \rangle$ expectation value $\Delta_\psi A := \sqrt{\langle A^2 \rangle_\psi - \langle A \rangle_\psi^2}$ rms deviation

QMIE

$A : \mathcal{H} \rightarrow \mathcal{H}$

linear, continuous, self-adjoint

$\cos \theta := |\langle \psi', \psi \rangle|$

absolute value of overlap

$\theta \in [0, \pi/2]$

$$|\langle A \rangle_\psi - \langle A \rangle_{\psi'}| \cos \theta \leq (\Delta_\psi A + \Delta_{\psi'} A) \sin \theta$$

$$|\langle \psi', \psi \rangle|^2 \leq \frac{(\Delta_\psi A + \Delta_{\psi'} A)^2}{(\langle A \rangle_\psi - \langle A \rangle_{\psi'})^2 + (\Delta_\psi A + \Delta_{\psi'} A)^2}$$

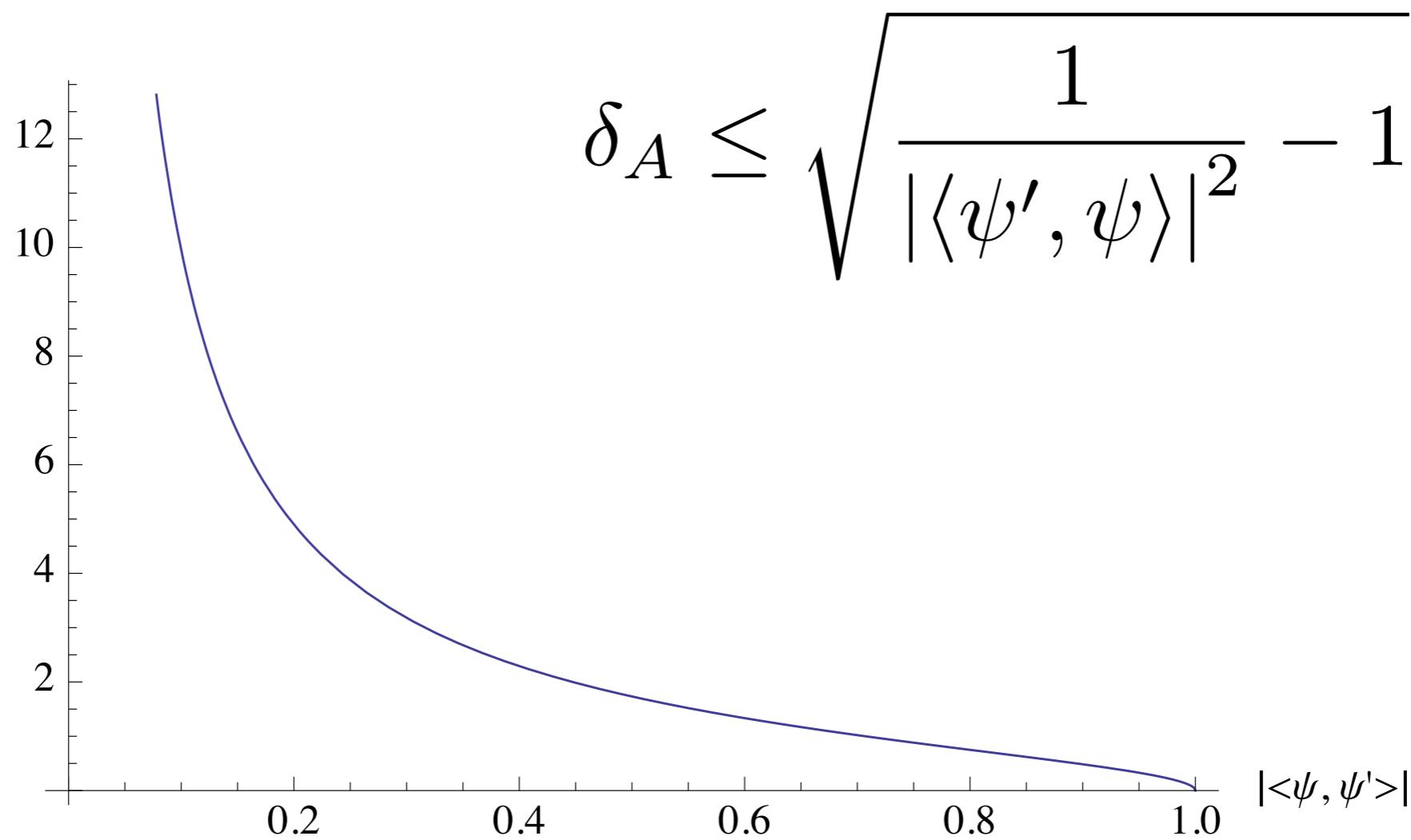
State Discrimination

QMIE generalises orthogonality of eigenvectors
to different eigenvalues of a self-adjoint operator

$$|\langle \psi', \psi \rangle|^2 \leq \frac{1}{1 + (\delta_A)^2} \quad \delta_A := \frac{|\langle A \rangle_\psi - \langle A \rangle_{\psi'}|}{\Delta_\psi A + \Delta_{\psi'} A}$$

$$\delta_A \leq \sqrt{\frac{1}{|\langle \psi', \psi \rangle|^2} - 1}$$

State Discrimination

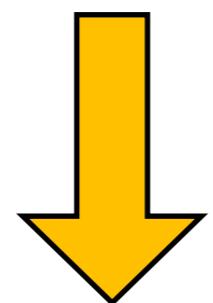


QMIE: Proof

auxiliary vector $\psi_A := A\psi - \langle A \rangle_\psi \psi$
 $\psi_A \perp \psi, |\psi_A| = \Delta_\psi A$

$$\langle \psi', A\psi \rangle = \langle \psi', \psi_A \rangle + \langle A \rangle_\psi \langle \psi', \psi \rangle$$

$$\langle A\psi', \psi \rangle = \langle \psi'_A, \psi \rangle + \langle A \rangle_{\psi'} \langle \psi', \psi \rangle$$



self-adjoint, rearrange
taking absolute values

$$|\langle A \rangle_\psi - \langle A \rangle_{\psi'}| |\langle \psi', \psi \rangle| = |\langle \psi'_A, \psi \rangle - \langle \psi', \psi_A \rangle|$$

QMIE: Proof

Estimate I: Triangle inequality

$$|\langle \psi'_A, \psi \rangle - \langle \psi', \psi_A \rangle| \leq |\langle \psi'_A, \psi \rangle| + |\langle \psi', \psi_A \rangle|$$

Estimate IIa: Decomposition

$$\psi = (\psi - \psi' \langle \psi', \psi \rangle) + \psi' \langle \psi', \psi \rangle$$

$$|\psi|^2 = |\langle \psi', \psi \rangle|^2 + |\psi - \psi' \langle \psi', \psi \rangle|^2$$

$$|\psi - \psi' \langle \psi', \psi \rangle|^2 = \sin^2 \theta$$

QMIE: Proof

Estimate IIb: Cauchy-Schwartz

$$|\langle \psi - \psi' \langle \psi', \psi \rangle, \psi'_A \rangle| \leq |\psi - \psi' \langle \psi', \psi \rangle| |\psi'_A|$$

$$|\langle \psi, \psi'_A \rangle| \leq (\sin \theta) (\Delta_{\psi'} A)$$

Analogously: $|\langle \psi', \psi_A \rangle| \leq (\sin \theta) (\Delta_\psi A)$

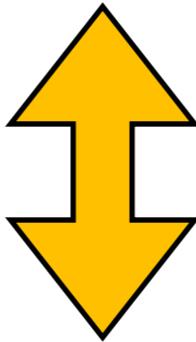
$$|\langle A \rangle_\psi - \langle A \rangle_{\psi'}| \cos \theta \leq (\Delta_\psi A + \Delta_{\psi'} A) \sin \theta$$

QMIE: Saturation

- Saturate triangle inequality
- Saturate Cauchy-Schwartz inequalities

QMIE: Saturation

$$|\langle A \rangle_{\psi} - \langle A \rangle_{\psi'}| \cos \theta = (\Delta_{\psi} A + \Delta_{\psi'} A) \sin \theta$$



$$A(\mathbb{C}\psi + \mathbb{C}\psi') \subseteq (\mathbb{C}\psi + \mathbb{C}\psi')$$

$$\langle \psi', A\psi \rangle = \lambda \langle \psi', \psi \rangle$$

$$\min\{\langle A \rangle_{\psi}, \langle A \rangle_{\psi'}\} \leq \lambda \leq \max\{\langle A \rangle_{\psi}, \langle A \rangle_{\psi'}\}$$

QMIE: Saturation

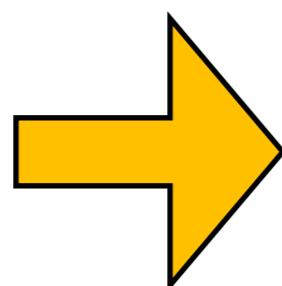
$$0 < \theta < \pi/2$$

from now on

(cases on the edge are trivial)

Cauchy-Schwartz: saturation iff involved
vectors are complex multiples of each other

$$\psi_A \in \mathbb{C} (\psi' - \psi \langle \psi, \psi' \rangle), \quad \psi'_A \in \mathbb{C} (\psi - \psi' \langle \psi', \psi \rangle)$$



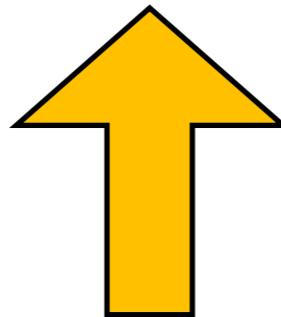
$$A\psi \in \mathbb{C}\psi + \mathbb{C}\psi'$$

$$A\psi' \in \mathbb{C}\psi + \mathbb{C}\psi'$$

QMIE: Saturation

$$\psi_A \parallel (\psi' - \psi \langle \psi, \psi' \rangle)$$

$$\psi'_A \parallel (\psi - \psi' \langle \psi', \psi \rangle)$$



$$A(\mathbb{C}\psi + \mathbb{C}\psi') \subseteq (\mathbb{C}\psi + \mathbb{C}\psi')$$

$$\psi_A \perp \psi, \psi'_A \perp \psi'$$

stabilised subspace is 2-dimensional

QMIE: Saturation

$$|\langle \psi'_A, \psi \rangle - \langle \psi', \psi_A \rangle| = |\langle \psi'_A, \psi \rangle| + |\langle \psi', \psi_A \rangle|$$

Triangle inequality: saturated iff complex numbers are real multiples of each other, i.e.

$$\begin{aligned} \alpha \langle \psi'_A, \psi \rangle + \beta \langle \psi', \psi_A \rangle &= 0 \\ (\alpha, \beta) &\in \mathbb{R}^2 \setminus 0 \end{aligned}$$

$$\begin{aligned} \langle \psi', A\psi \rangle &= \lambda \langle \psi', \psi \rangle \\ \lambda &= \frac{\alpha}{\alpha + \beta} \langle A \rangle_{\psi'} + \frac{\beta}{\alpha + \beta} \langle A \rangle_{\psi} \end{aligned}$$

Structure of A

Restricted to subspace stabilised by A now

$$\mathcal{H}_0 = \mathcal{C} \cdot \psi + \mathcal{C} \cdot \psi' \subset \mathcal{H}$$

Only look at restricted operator anymore

$$A|_{\mathcal{H}_0} = A_0 : \mathcal{H}_0 \rightarrow \mathcal{H}_0$$

$$\langle A \rangle_\psi \leq \langle A \rangle_{\psi'}$$

Structure of A

Choose an ONB $e = (e_1, e_2)$

$$\psi = e_1$$

$$\psi' = \cos \theta \cdot e_1 + \sin \theta \cdot e_2$$

One can ignore phases
(conditions are invariant)

Matrix representation of the operator

$$M(A_0, e) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$A_{11}, A_{22} \in \mathbb{R}, A_{12} = A_{21}^*$$

Structure of A

Use

$$\langle A \rangle_\psi \leq \frac{\langle \psi', A\psi \rangle}{\langle \psi', \psi \rangle} \leq \langle A \rangle_{\psi'}$$

to find

I.

$$\lambda = \frac{\langle \psi', A\psi \rangle}{\langle \psi', \psi \rangle} = A_{11} + \tan \theta \cdot A_{21}$$

$$A_{21} = A_{12}, A_{21}, A_{12} \in \mathbb{R}$$

Structure of A

2. At lower bound

$$\begin{aligned}A_{11} &\leq A_{11} + \tan \theta \cdot A_{21} \\A_{21} &\geq 0\end{aligned}$$

3. At upper bound

$$A_{22} \geq A_{11} + \tan \theta \cdot A_{21}$$

Structure of A

$$M(A_0, e) = \begin{pmatrix} s & a \\ a & s + b \cdot \tan \theta \end{pmatrix}$$

$$s, a, b \in \mathbb{R}, 0 \leq a \leq b$$

...tbc

- QMIE invariant under offset and scaling
- Normalise A to a spectrum of $\{I, -I\}$
- Reparametrise
- Put into context with other State Discrimination literature (Ivanovic, Peres)

References

- „Uses for a Quantum Master Inequality“,
G.N. Fleming, 2001
- „Fleming’s Quantum Master Inequality in
Spin-1/2-Systems“, Thesis, L. Ostermann, 2010
- Draft: „Saturation of Fleming’s QMIE“,
G. Grübl, L. Ostermann, 2011/12

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